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Analytical procedure for determining critical load of plates under variable boundary conditions

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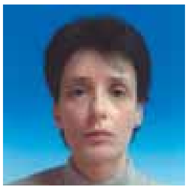

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Original scientific paper

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Analytical procedure for determining critical load of plates with variable boundary conditions

An analytical procedure for determining critical load, exerted on rectangular plates subjected to arbitrary external load under variable boundary conditions, is presented in the paper. The procedure used in critical load determination is based on the Ritz energy method. Because of complexity of mathematical model used, the case of locally distributed compressive stress is analyzed. The applicability and high level of accuracy of the method presented in the paper is proven through comparison of buckling coefficients and analysis results. This comparison is made using examples of plates with variable boundary conditions.

Key words:
elastic stability of plates, accurate stress function, combined boundary conditions, local load

Izvorni znanstveni rad

Olga Mijušković, Branislav Ćorić

Analitički postupak određivanja kritičnog opterećenja ploča različitih rubnih uvjeta

U radu je predstavljen analitički postupak za određivanje kritičnog opterećenja pravokutnih ploča različitih rubnih uvjeta pod djelovanjem proizvoljnog vanjskog opterećenja. Postupak određivanja kritičnog opterećenja temeljen je na Ritz-ovoj energijskoj metodi. S obzirom na složenost matematičkog modela, analiziran je slučaj lokalno raspodijeljenog tlačnog naprezanja. Na primjerima ploča različitih rubnih uvjeta, dokazana je primjenljivost i visoka točnost prikazane metode usporedbom koeficijenata izbočenja s rezultatima proračuna.

Ključne riječi:
elastična stabilnost ploča, točna funkcija naprezanja, mješoviti rubni uvjeti, lokalno opterećenje elastic

Wissenschaftlicher Originalbeitrag

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Analytisches Verfahren zur Feststellung der kritischen Belastung von Blechen mit verschiedenen Randbedingungen

In der Arbeit ist das analytische Verfahren zur Feststellung der kritischen Belastung von rechteckigen Blechen mit verschiedenen Randbedingungen unter der Wirkung von willkürlichen Außenbelastungen dargestellt. Das Verfahren zur Feststellung der kritischen Belastung basiert auf der Ritzschen Energiemethode. Hinsichtlich der Komplexität des mathematischen Modells wurde der Fall der lokal verteilten Druckbeanspruchung analysiert. Anhand von Beispielen von Blechen mit verschiedenen Randbedingungen, wurde die Anwendbarkeit und hohe Präzision der dargestellten Methode durch Vergleich der Knickkoeffizienten mit den Berechnungsresultaten aufgezeigt.

Schlüsselwörter:
Elastische Stabilität von Blechen, exakte Funktionsbeanspruchung, kombinierte Randbedingungen, Lokalbelastung

1. Introduction

The analytical procedure for determining the exact distribution of stress within a rectangular plate loaded in its own plane, which constitutes the basis of this paper, is based on the solution dating back to the 19th century. In fact, already in 1890 Mathieu [1] was the first to determine the function of stress for the concrete case of a rectangular plate loaded along the contour by an arbitrary compressive stress through superposition of two, the so called basic types of load (DEA and DEB), cf. Figure 1.

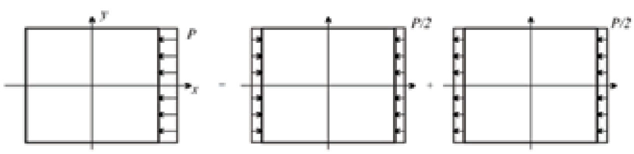


Figure 1. Division of load into cases of DEA and DEB

One of these cases (DEA) was used in the series of papers by Baker and Pavlović [2-5] during stability analysis of simply supported plates subjected to locally distributed compressive stress. However, several years later [6], more that a century after the original paper published by Mathieu, the same authors turned back to the basic problem, in order to find solution to the exact distribution of stress for a general case of a rectangular plate subjected to arbitrary external load. The essence lies in the fact that any load (normal and/or shear load) acting along the plate contour can be described by selected functions (odd and/or even with respect to coordinate axes), so that an overall solution is obtained by an appropriate combination of eight basic cases (Figure 2).

To enable a detailed description of the analytical procedure proposed in the paper, we will concentrate on the first basic case of load (DEA) in our analysis of plate behaviour at various edge conditions. According to literature, namely papers by Pavlović and Liu [7], only the examples of simply supported plates with different dimensions to deflection functions ratios, in form of double sine trigonometric series, have so far been analyzed, and they have shown a high level of adaptability to the real buckling pattern.

However, similar analyses have not as yet been made for plates with different boundary conditions. Considering that highly complex forms of external load occur in real-life steel structures, with exact stress distribution patterns, it is also highly significant to check whether the proposed functions properly describe different forms of buckling under any of the above mentioned basic forms of load. In this way, a high accuracy of results can be obtained during description of complex external influences, through superposition of two or more basic types of load.

2. Theoretical basis of the problem

Before resolving the selected case of load (DEA), it is indispensable to examine basic Mathieu's expressions based on in-plane elasticity equations, as his analysis and designations/marks partly differ from present-day approaches. In his paper, Mathieu expressed known equilibrium equations, without the action of volume forces, through displacements in form of:

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0 \qquad \Delta u=-\frac{1}{\varepsilon}\frac{d\nu}{dx} \qquad (1a)$$

$$\text{Mathieu} \Longrightarrow$$

$$\frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0 \qquad \Delta \nu=-\frac{1}{\varepsilon}\frac{d\nu}{dy} \qquad (1b)$$

where

Δ - Laplaceov operator
u, ν - displacements in the direction of *x* and *y*

$$\nu=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} \text{ - volume dilatation (change in volume)} \qquad (2)$$

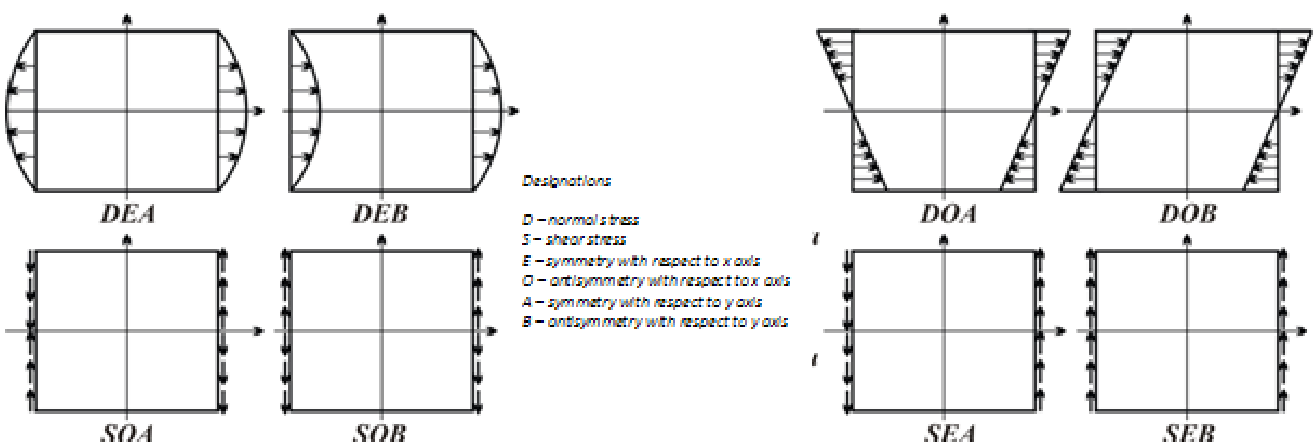


Figure 2. Eight basic types of load

$$\varepsilon = \frac{\mu}{\lambda + \mu} \quad \text{- constant defined by Lame coefficients.} \tag{3}$$

Using relatively simple mathematical operations, the equations from system (1) can be written as follows:

$$\Delta \nu = 0 \tag{4}$$

The Mathieu's approach to solving the problem of the plane elasticity theory starts by careful selection of functions for the value ν (4) in form of two Fourier series with unknown coefficients, by which the symmetry and antisymmetry of stress distribution has to be described in relation to x and y axes, respectively.

$$\nu = \nu_1 + \nu \tag{5}$$

In the next step, the function $F\,(F_1{+}F_2)$ is introduced based on the requirement that the following equation must be complied with:

$$\Delta F = -\frac{1}{\varepsilon}\nu \quad \Longrightarrow \quad \Delta F_1 = -\frac{1}{\varepsilon}\nu_1 \tag{6a}$$

$$\Delta F_2 = -\frac{1}{\varepsilon}\nu_2 \tag{6b}$$

Finally, once the displacements *u* and *v*, are defined:

$$u = \frac{dF}{dx} + \alpha\int \nu_1 dx \tag{7a}$$

$$\text{v} = \frac{dF}{dy} + \alpha\int \nu_2 dy \tag{7b}$$

where:

$\alpha = \frac{(\lambda + 2\mu)}{\mu}$ - constant expressed with Lamé parameters

Normal stresses N_1 and N_2 along x and y axes, and the shear stress T_3 in the x-y plane, are also defined.

$$N_1 = \lambda\nu + 2\mu\alpha\nu_1 + 2\mu\frac{d^2F}{dx^2} \tag{8a}$$

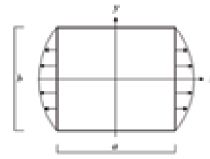
$$N_2 = \lambda\nu + 2\mu\alpha\nu_2 + 2\mu\frac{d^2F}{dy^2} \tag{8b}$$

$$T_3 = \mu\left[2\frac{d^2F}{dxdy} + \alpha\int\frac{d\nu_1}{dy}dx + \alpha\int\frac{d\nu_2}{dx}dy\right] \tag{8c}$$

The best and the simplest way to adequately explain the Mathieu's procedure is to analyze in detail one basic type of load (DEA) and to give appropriate explanations and comments.

3. Exact stress function under variable compressive stress (DEA)

The computation starts by introducing the external load expression in form of a Fourier series:



$$f(y) = A_0 + \sum_n A_n \cos ny$$

as the DEA case is characterized by symmetry with respect to both axes. In addition, the selection of functions for components ν_1 and ν_2 is contingent upon existence of double symmetry.

$$\nu_1 = B_0 + \sum_n B_n \cosh(nx)\cos ny \quad n = \frac{2q\pi}{b} \quad q = 1,2,3,\ldots \tag{9a}$$

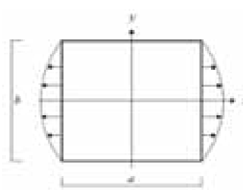
$$\nu_2 = \beta_0 + \sum_m \beta_m \cosh(my)\cos mx \quad m = \frac{2p\pi}{a} \quad p = 1,2,3,\ldots \tag{9b}$$

Just like the dilatation is shown by superposition of two functions, the value *F* is also divided into components F_1 and F_2, which are the solution to the partial differential equation (6).

$$F_1 = -\frac{1}{2\varepsilon}B_0x^2 - \frac{1}{2\varepsilon}\sum_n \frac{1}{n}B_n\, xe(nx)\cos ny + \sum_n H_n E(nx)\cos ny \tag{10a}$$

$$F_2 = -\frac{1}{2\varepsilon}\beta_0y^2 - \frac{1}{2\varepsilon}\sum_m \frac{1}{m}\beta_m\, ye(my)\cos mx + \sum_m G_m E(my)\cos mx \tag{10b}$$

To enable shorter writing of very complex expressions, appropriate abbreviations are introduced (e.g. *E(•) = cosh(•)*, or *e(•) = sinh(•)*) Their meaning will be explained in the course of the derivation.
And finally, by introducing the functions F_1 and F_2 into expressions for stress (8), the solution of the problem is reduced to determination of four unknown groups of coefficients B_n, β_m, H_n and *G* from available boundary conditions. There are in principle eight conditions (two on each contour) for the case of rectangular plate, but they are reduced to indispensable four, because of properties of the introduced dilatation functions. In the DEA example, these are:



DEA Boundary conditions:

$$N_1 = f(y), \quad x = \pm\frac{a}{2} \tag{11a}$$

$$N_2 = 0, \quad y = \pm\frac{b}{2} \tag{11b}$$

$$T_3 = 0, \quad x = \pm\frac{a}{2}, \quad y = \pm\frac{b}{2} \tag{11c}$$

Thus, the solving procedure continues with selection of two boundary conditions by which the direct dependence between appropriate groups of unknown coefficients is defined, while

the remaining two reduce the problem to an infinite system of linear equations which contain parameters of external load.
By using the original Mathieu procedure for solving this problem, the group of coefficients H_n and G_m can be defined provided that shear stress values at the contour are equal to zero (11c).

$$H_n = B_n\left[-\frac{1}{2n^2}+\frac{a}{4n\varepsilon}\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}\right] \tag{12a}$$

$$G_m = \beta_m\left[-\frac{1}{2m^2}+\frac{b}{4m\varepsilon}\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}\right] \tag{12b}$$

The second group of boundary conditions, which is related to distribution of normal stresses, leads the definition of the equation system towards unknown coefficients B_n and $\beta_{m'}$ which unambiguously define the displacement components and the change in stress within the plate for the analyzed case of load.

$$B_n = \frac{A_n}{(\lambda+\mu)\tau(\frac{1}{2}na)} - \frac{8n^2\cos\frac{1}{2}nb}{b\tau(\frac{1}{2}na)}\sum_m \beta_m \frac{me(\frac{1}{2}mb)\cos\frac{1}{2}ma}{(m^2+n^2)^2} \tag{13a}$$

$$\beta_m = -\frac{8m^2\cos\frac{1}{2}ma}{a\tau(\frac{1}{2}mb)}\sum_n B_n \frac{ne(\frac{1}{2}na)\cos\frac{1}{2}nb}{(m^2+n^2)^2} \tag{13b}$$

The dependence between values B_n and $\beta_{m'}$ and between external load parameters, is defined in these expressions. And finally, an infinite system of equations (14) is obtained by replacing in the expression β_m with B_n and vice versa, and by multiple introduction of appropriate sums, their combinations of positions, and by reduction to dimensionless form. Naturally, in practical examples we are forced to introduce a finite number of members in a series, and this number must be defined very carefully from the standpoint of convergence and accuracy.

$$B_q = \frac{A_q}{(\lambda+\mu)\tau(q\pi\varphi)} + \frac{16\varphi^4 q^2(-1)^q}{(\lambda+\mu)\pi^2\tau(q\pi\varphi)}\sum_{q'} q' A_{q'}(-1)^{q'}\Psi(q'\pi\varphi)$$
$$\times\{\Lambda_1(q,q') + (16\varphi^4/\pi^2)\Lambda_3(q,q') + (16\varphi^4/\pi^2)^2\Lambda_5(q,q') + \cdots\} \tag{14a}$$

$$\beta_p = -\frac{4\varphi p^2(-1)^p}{(\lambda+\mu)\pi\tau(p\pi/\varphi)}\sum_q qA_q(-1)^q\Psi(q\pi\varphi)$$
$$\times\{\Lambda_0(p,q) + (16\varphi^4/\pi^2)\Lambda_2(p,q) + (16\varphi^4/\pi^2)^2\Lambda_4(p,q) + \cdots\} \tag{14b}$$

To simplify the expressions, the following abbreviations are introduced

$$\tau(x) = E(x) + x/e(x) \quad i \quad \Psi(x) = e(x)/\tau(x) \tag{15}$$

while Λ are dimensionless functions of form:

$$\Lambda_0(p,q) = \frac{1}{(p^2+\phi^2q^2)^2} \tag{16a}$$

$$\Lambda_1(q,q') = \sum_p \frac{p^3\Psi(p\pi/\phi)}{(p^2+\phi^2q'^2)^2}\Lambda_0(p,q) \tag{16b}$$

$$\Lambda_2(p,q) = \sum_{q'} \frac{q'^3\Psi(q'\pi\phi)}{(p^2+\phi^2q'^2)^2}\Lambda_1(q,q') \tag{16c}$$

$$\Lambda_3(q,q') = \sum_p \frac{p^3\Psi(p\pi/\phi)}{(p^2+\phi^2q'^2)^2}\Lambda_2(p,q) \tag{16d}$$

$$\Lambda_4(p,q) = \sum_{q'} \frac{q'^3\Psi(q'\pi\phi)}{(p^2+\phi^2q'^2)^2}\Lambda_3(q,q') \tag{16e}$$

. . .

Expressions for displacements, but only as the functions of coordinates x and y, are obtained from the expression (7) as follows:

$$u = B_0x + \sum_n B_n\left[\left(\frac{\alpha}{2n}+\frac{a}{4\varepsilon}\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}\right)e(nx) - \frac{xE(nx)}{2\varepsilon}\right]\cos ny$$
$$+\sum_m \beta_m\left[\left(\frac{1}{2m}-\frac{b}{4\varepsilon}\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}\right)E(my) + \frac{ye(my)}{2\varepsilon}\right]\sin mx \tag{17a}$$

$$v = \beta_0y + \sum_n B_n\left[\left(\frac{1}{2n}-\frac{a}{4\varepsilon}\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}\right)E(nx) + \frac{xe(nx)}{2\varepsilon}\right]\sin ny$$
$$+\sum_m \beta_m\left[\left(\frac{\alpha}{2m}+\frac{b}{4\varepsilon}\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}\right)e(my) - \frac{yE(my)}{2\varepsilon}\right]\cos mx \tag{17b}$$

Expressions for the exact stress functions, for the case of the load under study (DEA), are obtained in accordance with expressions (8):

$$N_1 = A_0 + (\lambda+\mu)\sum_n B_n\left[\left(1+\frac{na}{2}\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}\right)E(nx) - nxe(nx)\right]\cos ny$$
$$+(\lambda+\mu)\sum_m \beta_m\left[\left(1-\frac{mb}{2}\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}\right)E(my) + my\,e(my)\right]\cos mx \tag{18a}$$

$$N_2 = (\lambda+\mu)\sum_n B_n\left[\left(1-\frac{na}{2}\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}\right)E(nx) + nxe(nx)\right]\cos ny$$
$$+(\lambda+\mu)\sum_m \beta_m\left[\left(1+\frac{mb}{2}\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}\right)E(my) - my\,e(my)\right]\cos mx \tag{18b}$$

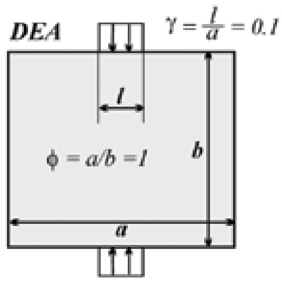
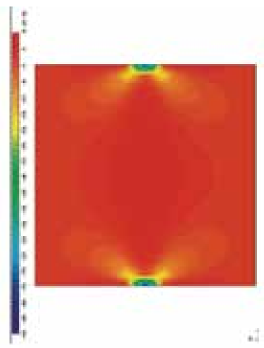
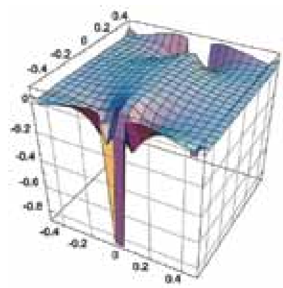
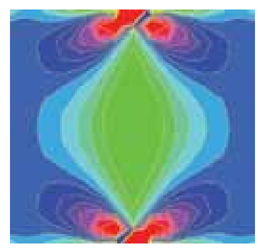
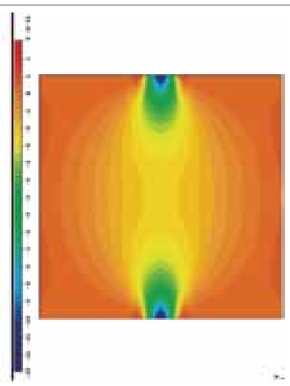
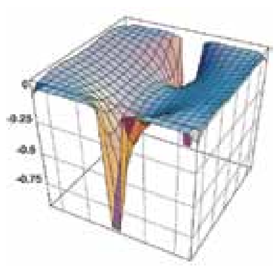
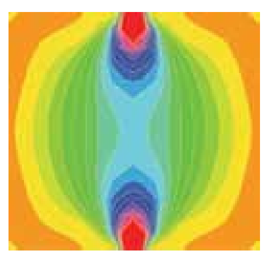
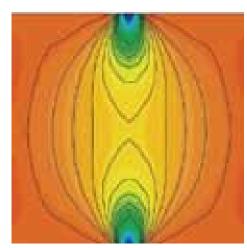
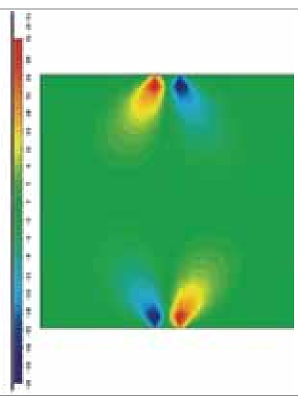
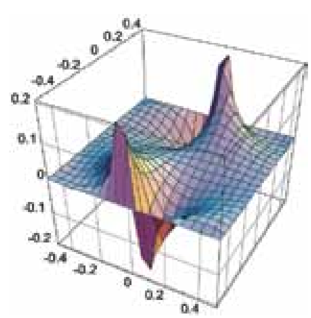
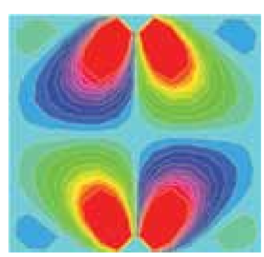
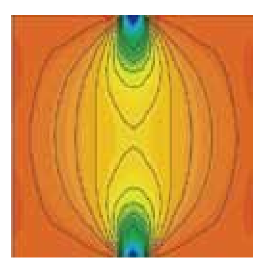
$$T_3 = (\lambda+\mu)\sum_n B_n\left(-\frac{na}{2}\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}e(nx) + nxE(nx)\right)\sin ny$$
$$+(\lambda+\mu)\sum_m \beta_m\left(-\frac{mb}{2}\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}e(my) + myE(my)\right)\sin mx \tag{18c}$$

And finally, to present the solution in form of a table, stress distributions obtained analytically and using the programs FINEL and ANSYS (based on the finite-element method) are presented side by side in Table 1.
The presented diagrams show that the stress distribution at points subjected to load can hardly be considered uniform, which used to be the most frequent assumption, and also the biggest error, in the analysis of similar problems as made prior to the introduction of exact solutions.

4. Formulation of stability problem

The problem of elastic stability of rectangular plates, of various boundary conditions, is considered using the Ritz energy procedure,

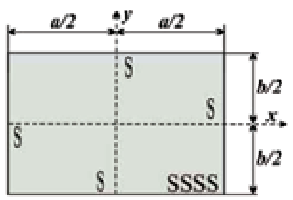
Table 1. Stress distribution within the plate, defined by analytical procedure and by computation

Plate with combined boundary conditions for dimension relationships ϕ = a/b = 1,0			
			
	Computation (Finel)	Analytical solution	Check-overlap
σ_x		 	
	Computation (Finel)	Analytical solution	Check-overlap
σ_y		 	
	Computation (Finel)	Analytical solution	Check-overlap
τ_{xy}		 	

in the scope of which the energy of deformation of a bent plate is defined in a traditional way, while the real stress distribution from the Mathieu's theory of elasticity is introduced during determination of the external force action. Elastic buckling coefficients can be defined very accurately using the exact distribution of stress for any arbitrary external load, and double Fourier series for adequate description of appropriate forms of buckling. The analytic procedure is presented using as example four different types of plates of mixed boundary conditions (SSSS, SCSC, CSCS and CCCC) under a variable compressive stress (DEA). Of course, adequate programs, based on the finite-element method (FINEL and ANSYS), are used for checking the buckling coefficients, as no similar analytical solutions have so far been presented in literature.

4.1. Buckling functions adopted in the analysis

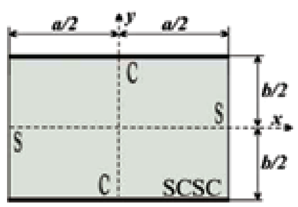
For the selected plate and load examples (Figures 3 to 6), we have adopted deflection functions in form of a double Fourier series (19–22) by which we can meet, member by member, the selected boundary conditions, and which have been proven to approximate the deformed configuration quite well, for a very wide spectrum of plate dimension relationships.



Example 1
edges x = ± a/2 simply supported (S)
edges y = ± b/2 simply supported (S)

Figure 3. Simply supported plate (Example 1:SSSS)

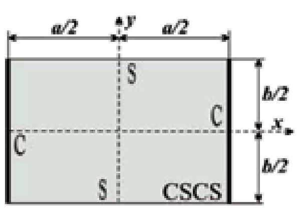
$$w=\sum_{m=1}^{s}\sum_{n=1}^{s}W_{mn}\sin\frac{m\pi}{a}\left(x+\frac{a}{2}\right)\sin\frac{n\pi}{b}\left(y+\frac{b}{2}\right) \tag{19}$$



Example 2
edges x = ± a/2 simply supported (S)
edges y = ± b/2 simply constrained (S)

Figure 4. Plate with combined boundary conditions (Example 2:SCSC)

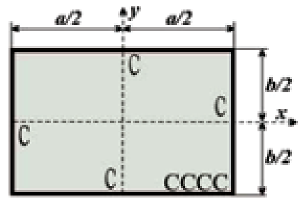
$$w=\sum_{m=1}^{s}\sum_{n=1}^{s}W_{mn}\sin\frac{m\pi}{a}\left(x+\frac{a}{2}\right)\left(\cos\frac{(n-1)\pi}{b}\left(y+\frac{b}{2}\right)-\cos\frac{(n+1)\pi}{b}\left(y+\frac{b}{2}\right)\right) \tag{20}$$



Example 3
edges x = ± a/2 constrained (S)
edges y = ± b/2 simply supported (S)

Figure 5. Plate with combined boundary conditions (Example 3:SCSC)

$$w=\sum_{m=1}^{s}\sum_{n=1}^{s}W_{mn}\left(\cos\frac{(m-1)\pi}{a}\left(x+\frac{a}{2}\right)-\cos\frac{(m+1)\pi}{a}\left(x+\frac{a}{2}\right)\right)\sin\frac{n\pi}{b}\left(y+\frac{b}{2}\right) \tag{21}$$



Example 4
edges x = ± a/2 constrained (S)
edges y = ± b/2 constrained (S)

Figure 6. Constrained plate (Example 4:CCCC)

$$w=\sum_{m=1}^{s}\sum_{n=1}^{s}W_{mn}\left(\cos\frac{(m-1)\pi}{a}\left(x+\frac{a}{2}\right)-\cos\frac{(m+1)\pi}{a}\left(x+\frac{a}{2}\right)\right)\cdot \left(\cos\frac{(n-1)\pi}{b}\left(y+\frac{b}{2}\right)-\cos\frac{(n+1)\pi}{b}\left(y+\frac{b}{2}\right)\right) \tag{22}$$

4.2. Plate bending strain energy

When defining the total potential energy of a system in the scope of the Ritz energy procedure, the first step is to define strain energy at plate bending, and this in a usual traditional way:

$$U=\frac{1}{2}D\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\left[\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2-2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right]dxdy \tag{23}$$

where D is flexural stiffness.

4.3. Action of external forces

The part of the system's potential energy that is related to the action of external forces is represented by the expression (24), which now contains designations of the exact stress distribution N_1, N_2, and T_3 from the Mathieu's theory of elasticity (18):

$$V=-\frac{t}{2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\left[N_1\left(\frac{\partial w}{\partial x}\right)^2+N_2\left(\frac{\partial w}{\partial y}\right)^2+2T_3\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]dxdy \tag{24}$$

Therefore, by introducing the exact stress functions the expression which takes into account the action of external forces becomes much more complicated. This is the fundamental difference when compared to other stability analyses where plates are not exclusively simply supported along all edges. In order to simplify the computer program burdened by the presence of many integrals containing various combinations of hyperbolic and trigonometric functions, included in deflection and stress functions, it was indispensable to simply the expression (24) and consider contribution of individual members (25, 25a, 25b, 25c):

$$V=V_1+V_2+V_3 \tag{25}$$

where:

$$V_1=-\frac{t}{2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}N_1\left(\frac{\partial w}{\partial x}\right)^2 dxdy \tag{25a}$$

$$V_2=-\frac{t}{2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}N_2\left(\frac{\partial w}{\partial y}\right)^2 dxdy \tag{25b}$$

$$V_3 = -t \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} T_3 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} dxdy \tag{25c}$$

4.4. Reduction to the problem of eigenvalues

After definition of the plate-bending strain energy, U, and the external force action, V, the total potential energy of the system can be written as follows:

$$\Pi = U + V \tag{26}$$

The condition (27) is obtained from the principle of minimum potential energy:

$$\frac{\partial \Pi}{\partial W_{mn}} = \frac{\partial U}{\partial W_{mn}} + \frac{\partial V}{\partial W_{mn}} \tag{27}$$

which can in principle be reduced to the system of m·n homogeneous linear equations according to unknown coefficients W_{mn}. The existence of nontrivial solution, i.e. of the requirement that the determinant of the system is equal to zero, is reduced to the traditional problem of eigenvalues, in the scope of which only the lowest eigenvalue is of practical significance to us, and it represents the critical load we are looking for. Of course, due to extensiveness of the procedure for approximation of stress and deflection functions, which is directly dependent on the number of members in the series, the use of an appropriate computer program (MATHEMATICA) is considered indispensable.

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5. Examples and results

As already indicated, in order to prove usability of the analytical procedure used in the paper to solve elastic stability problems, four different plate types: SSSS, SCSC, CSCS and CCCC (Figures 3 to 6), are considered. They are influenced by locally distributed stresses (DEA), for which no analytical results exist in the literature (Figure 7). All results for different dimension relationships (ϕ = 0,3 – 2,0), and for the influence of the local compressive stress (γ= 0,0 – 1,0), are presented tabularly (Tables 2 to 5). In addition to buckling coefficients obtained by the derived analytical procedure, they also include calculation results obtained using the FINEL and ANSYS software, based on the finite-element method.

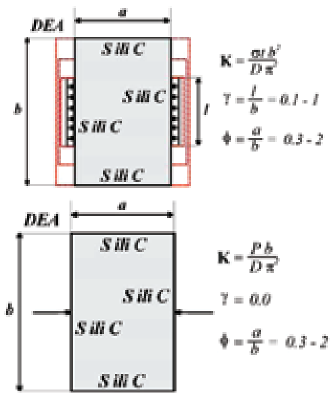


Figure 7. Analysed example

Table 2. Buckling coefficient values for "SSSS" plate

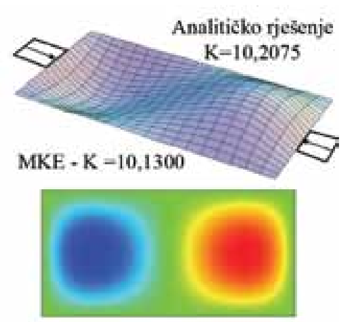
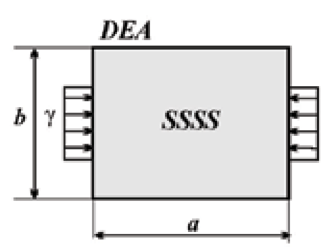
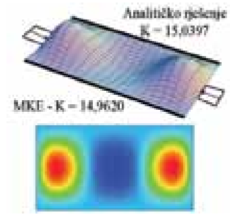
Plate SSSS ϕ = 0,3 – 2,0									
	ϕ = 0.3	ϕ =0.5	ϕ =0.7	ϕ =0.9	ϕ =1.0	ϕ =1.5	ϕ =2.0	Results	Example ϕ = 2,0; γ = 0,3
γ = 0.0	48.5579	30.0726	24.7160	24.8468	26.0093	29.1098	29.0185	Analytical sol.	
	48.0267	29.8260	24.4957	24.5922	25.7770	28.8980	28.7950	MKE (FI, AN)	
	(-1.094)	(-0.820)	(-0.891)	(-0.905)	(-0.893)	(-0.728)	(-0.770)	Difference (%)	
γ = 0.1	50.1069	30.4570	24.9336	25.0023	26.1936	29.3081	29.2074	Analytical sol.	
	49.5980	30.2120	24.7140	24.7770	25.9600	29.0970	28.9840	MKE (FI, AN)	
	(-1.016)	(-0.804)	(-0.881)	(-0.901)	(-0.892)	(-0.720)	(-0.765)	Difference (%)	
γ = 0.3	20.1632	11.1184	8.8846	8.8212	9.2096	10.2697	10.2075	Analytical sol.	
	19.9840	11.0290	8.8058	8.7418	9.1280	10.1960	10.1300	MKE (FI, AN)	
	(-0.889)	(-0.804)	(-0.887)	(-0.900)	(-0.886)	(-0.718)	(-0.759)	Difference (%)	
γ = 0.5	15.3162	7.7388	6.0090	5.8632	6.0748	6.7109	6.6257	Analytical sol.	
	15.1950	7.6753	5.9548	5.8102	6.0210	6.6628	6.5761	MKE (FI, AN)	
	(-0.791)	(-0.821)	(-0.902)	(-0.904)	(-0.886)	(-0.717)	(-0.749)	Difference (%)	
γ = 0.7	13.7073	6.6273	5.0489	4.7722	4.8794	5.3243	5.1744	Analytical sol.	
	13.6020	6.5708	4.9728	4.7290	4.8370	5.2863	5.1370	MKE (FI, AN)	
	(-0.768)	(-0.852)	(-0.918)	(-0.905)	(-0.869)	(-0.714)	(-0.723)	Difference (%)	
γ = 0.9	13.2112	6.2858	4.6330	4.2385	4.2457	4.6016	4.3331	Analytical sol.	
	13.1050	6.2305	4.5899	4.2005	4.2090	4.5695	4.3038	MKE (FI, AN)	
	(-0.804)	(-0.880)	(-0.930)	(-0.896)	(-0.864)	(-0.697)	(-0.676)	Difference (%)	
γ = 1.0	13.2010	6.2500	4.5308	4.0446	4.0000	4.3403	4.0000	Analytical sol.	
	13.0950	6.1947	4.4887	4.0086	3.9660	4.3106	3.9740	MKE (FI, AN)	
	(-0.803)	(-0.885)	(-0.929)	(-0.890)	(-0.850)	(-0.684)	(-0.648)	Difference (%)	

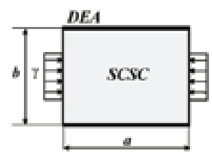
Table 3. Buckling coefficient values for slabs SCSC and CSCS

Plate SCSC ϕ = 0.3 – 2,0

	ϕ = 0.3	ϕ =0.5	ϕ=0.7	ϕ =0.9	ϕ =1.0	ϕ =1.5	ϕ =2.0	Results
γ = 0.0	48.7433	33.1740	34.0400	41.1378	39.4680	40.6440	41.0340	Analytical sol.
	48.6300	33.1200	33.9914	41.0900	39.4210	40.4030	40.7920	MKE (FI, AN)
	(-0.232)	(-0.163)	(-0.143)	(-0.116)	(-0.119)	(-0.593)	(-0.590)	Difference (%)
γ = 0.1	50.3111	33.6731	34.4509	41.6785	39.9466	41.1357	41.5313	Analytical sol.
	50.2000	33.6200	34.2360	41.4510	39.7330	40.9010	41.2960	MKE (FI, AN)
	(-0.221)	(-0.158)	(-0.624)	(-0.546)	(-0.535)	(-0.571)	(-0.567)	Difference (%)
γ = 0.3	20.3074	12.4985	12.5559	15.2077	14.4908	14.9050	15.0397	Analytical sol.
	20.2667	12.4800	12.4800	15.1310	14.4190	14.8270	14.9620	MKE (FI, AN)
	(-0.201)	(-0.148)	(-0.604)	(-0.504)	(-0.495)	(-0.523)	(-0.517)	Difference (%)
γ = 0.5	15.5533	8.9588	8.7929	10.5539	9.9820	10.2147	10.2723	Analytical sol.
	15.5267	8.9456	8.7414	10.5030	9.9347	10.1650	10.2240	MKE (FI, AN)
	(-0.171)	(-0.147)	(-0.586)	(-0.482)	(-0.474)	(-0.486)	(-0.470)	Difference (%)
γ = 0.7	14.1556	7.9297	7.5893	8.9950	8.4330	8.4768	8.4607	Analytical sol.
	14.1317	7.9177	7.5457	8.9526	8.3942	8.4396	8.4253	MKE (FI, AN)
	(-0.168)	(-0.151)	(-0.574)	(-0.472)	(-0.460)	(-0.439)	(-0.418)	Difference (%)
γ = 0.9	13.8716	7.6874	7.1305	8.2673	7.8284	7.5142	7.4138	Analytical sol.
	13.8481	7.6756	7.0897	8.2300	7.7928	7.4854	7.3876	MKE (FI, AN)
	(-0.169)	(-0.153)	(-0.572)	(-0.451)	(-0.455)	(-0.383)	(-0.353)	Difference (%)
γ = 1.0	13.9044	7.6927	7.0021	7.8583	7.6927	7.1172	6.9729	Analytical sol.
	13.8822	7.6812	6.9624	7.8234	7.6586	7.0923	6.9511	MKE (FI, AN)
	(-0.160)	(-0.149)	(-0.567)	(-0.438)	(-0.443)	(-0.350)	(-0.312)	Difference (%)

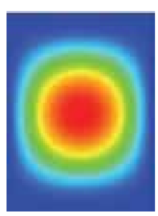
Example ϕ = 2,0; γ = 0,3



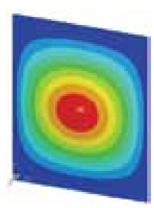


Oblik izbočivanja SCSC (slučaj ϕ= 0,7 i γ = 0,3)

MKE – FINEL (K = 12.4800)



MKE – ANSYS (K = 12.5401)



Analytical solution (K = 12.5559)

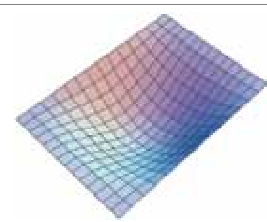


Plate CSCS ϕ = 0.3 – 2,0

	ϕ = 0.3	ϕ=0.5	ϕ =0.7	ϕ =0.9	ϕ =1.0	ϕ =1.5	ϕ =2.0	Results
γ = 0.0	136.527	81.3980	58.0843	48.8844	47.3130	47.6780	43.952	Analytical sol.
	136.390	81.3460	58.0414	48.8433	47.2680	47.4730	43.741	MKE (FI, AN)
	(-0.100)	(-0.064)	(-0.074)	(-0.084)	(-0.095)	(-0.430)	(-0.480)	Difference (%)
γ = 0.1	143.156	82.9067	58.6814	49.2515	47.6257	47.8902	44.0682	Analytical sol.
	143.111	82.8760	58.4730	49.0760	47.4510	47.6950	43.8630	MKE (FI, AN)
	(-0.031)	(-0.037)	(-0.355)	(-0.356)	(-0.367)	(-0.408)	(-0.466)	Difference (%)
γ = 0.3	61.9592	30.9996	20.9747	17.3066	16.6288	16.4781	14.9578	Analytical sol.
	61.9370	30.9907	20.9060	17.2480	16.5710	16.4160	14.8940	MKE (FI, AN)
	(-0.036)	(-0.029)	(-0.328)	(-0.339)	(-0.347)	(-0.377)	(-0.426)	Difference (%)
γ = 0.5	50.7689	22.0930	14.1475	11.3570	10.7786	10.4280	9.2197	Analytical sol.
	50.7511	22.0864	14.1010	11.3190	10.7420	10.3900	9.1821	MKE (FI, AN)
	(-0.035)	(-0.030)	(-0.329)	(-0.335)	(-0.340)	(-0.364)	(-0.408)	Difference (%)

Example ϕ = 2,0; γ = 0,3

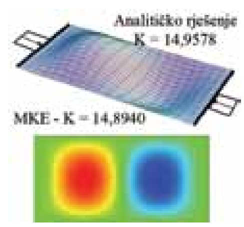


Table 3. - continuation - Buckling coefficient values for slabs CSCS

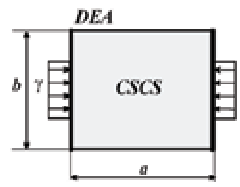
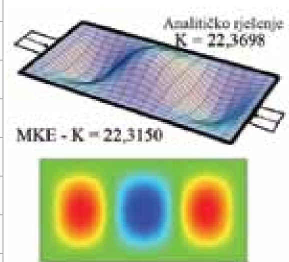
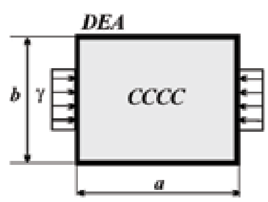
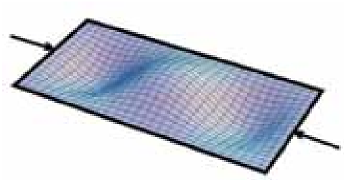
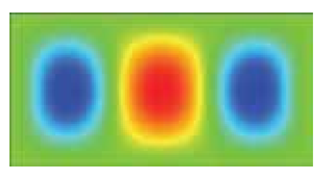
γ = 0.7	47.5762	19.1541	11.7280	9.0741	8.4494	7.8059	6.7607	Analytical sol.	
	47.5587	19.1469	11.6880	9.0437	8.4205	7.7774	6.7335	MKE (FI, AN)	
	(-0.037)	(-0.038)	(-0.341)	(-0.335)	(-0.342)	(-0.365)	(-0.402)	Razlika (%)	
γ = 0.9	46.4395	18.2268	10.7562	7.9292	7.1948	6.0301	5.3557	Analytical sol.	
	46.4173	18.2182	10.7180	7.9018	7.1703	6.0082	5.3341	MKE (FI, AN)	
	(-0.048)	(-0.047)	(-0.355)	(-0.345)	(-0.340)	(-0.363)	(-0.403)	Razlika (%)	
γ = 1.0	46.5122	18.1874	10.5300	7.5422	6.7432	5.3748	4.8472	Analytical sol.	
	46.4922	18.1800	10.4920	7.5160	6.7196	5.3551	4.8275	MKE (FI, AN)	
	(-0.043)	(-0.041)	(-0.361)	(-0.347)	(-0.350)	(-0.366)	(-0.406)	Razlika (%)	

Table 4. Buckling coefficient values for "CCCC" plate

Plate CCCC ϕ = 0.3 – 2,0									
	ϕ = 0.3	ϕ =0.5	ϕ =0.7	ϕ =0.9	ϕ =1.0	ϕ =1.5	ϕ =2.0	Results	Example ϕ = 2,0; γ = 0,3
γ = 0.0	136.507	81.9980	63.9057	64.4689	67.1260	63.8240	64.5070	Analytical sol.	
	136.390	81.9460	63.8686	64.4322	67.0660	63.6270	64.2230	MKE (FI, AN)	
	(-0.086)	(-0.063)	(-0.058)	(-0.057)	(-0.089)	(-0.309)	(-0.440)	Difference (%)	
γ = 0.1	143.156	83.5588	64.6723	65.0843	67.7444	64.2124	64.8332	Analytical sol.	
	143.111	83.5280	64.4790	64.9180	67.5360	64.0360	64.5890	MKE (FI, AN)	
	(-0.031)	(-0.037)	(-0.299)	(-0.255)	(-0.308)	(-0.275)	(-0.377)	Difference (%)	
γ = 0.3	61.9592	31.3774	23.3818	23.1485	24.0774	22.3021	22.3698	Analytical sol.	
	61.9370	31.3693	23.3210	23.0960	24.0120	22.2540	22.3150	MKE (FI, AN)	
	(-0.036)	(-0.026)	(-0.260)	(-0.227)	(-0.272)	(-0.217)	(-0.245)	Difference (%)	
γ = 0.5	50.7778	22.5921	16.0615	15.4186	16.1085	14.2605	14.1285	Analytical sol.	
	50.7600	22.5864	16.0210	15.3860	16.0660	14.2320	14.1000	MKE (FI, AN)	
	(-0.035)	(-0.025)	(-0.252)	(-0.211)	(-0.264)	(-0.200)	(-0.202)	Difference (%)	
γ = 0.7	47.6841	19.9012	13.5669	12.4490	12.7037	10.9178	10.6209	Analytical sol.	
	47.6667	19.8949	13.5320	12.4230	12.6770	10.8970	10.6000	MKE (FI, AN)	
	(-0.036)	(-0.032)	(-0.257)	(-0.209)	(-0.210)	(-0.191)	(-0.197)	Difference (%)	
γ = 0.9	46.8778	19.2366	12.6253	10.9173	10.7885	9.0356	8.6047	Analytical sol.	
	46.8580	19.2289	12.5920	10.8940	10.7660	9.0180	8.5880	MKE (FI, AN)	
	(-0.042)	(-0.040)	(-0.264)	(-0.213)	(-0.209)	(-0.195)	(-0.194)	Difference (%)	
γ = 1.0	47.0944	19.3406	12.4458	10.3857	10.0760	8.3521	7.8686	Analytical sol.	
	47.0756	19.3340	12.4140	10.3650	10.0560	8.3362	7.8536	MKE (FI, AN)	
	(-0.040)	(-0.034)	(-0.255)	(-0.202)	(-0.198)	(-0.190)	(-0.191)	Difference (%)	
Oblik izbočivanja CCCC (slučaj ϕ = 2,0 i γ = 0)	Analytical solution K= 64,507							MKE (FINEL) K= 64,223	
									

To enable proper evaluation of the analytical solutions presented in the paper, it is important to take note of some limitations that had to be introduced for practical reasons. In fact, buckling coefficients presented in the tables were obtained by introducing 20 members for the deflection function series, and 60 members for the stress function series. For stress functions, an absolute stability of the solution is obtained with 40 or more members, while in case of deflection functions even more members would sometimes be needed, depending on boundary conditions and load. By limiting the number of members, we have obtained the solution that is somewhat higher than the exact solution which would, of course, be attainable if we were able to introduce an infinite number of members. However, due to extreme complexity of the procedure, and great number of integrals to be solved, the analysis was made with 20 members of the deflection function series, for all plate dimensions and boundary conditions, using the above mentioned normal load.

After a detailed analysis of the results, the accuracy of assumptions was realistically evaluated, and this not only for the Ritz method, but also for the design parameters included in the method, which directly influence both convergence and level of accuracy of the solution.

6. Conclusion

After a detailed analysis of results (Tables 2 to 5), it can easily be noted that the analytical solution manifests a good behaviour in the studied range of plate dimensions (ϕ = 0.3 – 2.0) and load (γ = 0.0 – 1.0). Buckling coefficient values for four selected plate types (SSSS, SCSC, CSCS and CCCC) under locally distributed compressive stress, show maximum deviation with respect to values gained using the finite-element method of no more than 1.0 percent (SSSS ϕ = 0.3 and γ = 0.0). Nevertheless, this difference should be taken as relative because, for the same example, and using the ANSYS program, the value obtained (K = 48.482) deviates by only 0.16 percent from the analytical solution. In other words, solutions obtained by the finite-element method (FINEL, ANSYS) depend on the network density and on the adopted type of final element, and have their limitations as to the level of accuracy. This is also true for the analytical procedure presented in this paper, due to limited number of members in the series. Finally, it can be concluded that the number of members in the series of deflection and stress functions is sufficient for the analysed cases of plates with mixed boundary conditions, subjected to variable compressive load. Obviously, in cases with more complex types of load, which were also subjected to detailed verification, it is indispensable to pay special attention to proper selection of the type and number of members in the series, and to stress and deflection functions, which undoubtedly directly influences the level of accuracy of solutions, although not the applicability of the analytical procedure presented in the paper.

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